The derivation of Poiseuille's law: heuristic and explanatory considerations

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Abstract

This paper illustrates how an experimental discovery can prompt the search for a theoretical explanation and also how obtaining such an explanation can provide heuristic benefits for further experimental discoveries. The case considered begins with the discovery of Poiseuille's law for steady fluid flow through pipes. The law was originally supported by careful experiments, and was only later explained through a derivation from the more basic Navier-Stokes equations. However, this derivation employed a controversial boundary condition and also relied on a contentious approach to viscosity. By comparing two editions of Lamb's famous *Hydrodynamics* textbook, I argue that explanatory considerations were central to Lamb's claims about this sort of fluid flow. In addition, I argue that this treatment of Poiseuille's law played a heuristic role in Reynolds' treatment of turbulent flows, where Poiseuille's law fails to apply.

1. Introduction. Often in the history of science some feature of a phenomenon is identified through experiment long before it is possible to explain that feature. According to the influential account offered by Bogen and Woodward, a phenomenon is a repeatable type of event, state, or process [Bogen and Woodward, 1988]. An experiment can provide data that is good evidence for some feature of some phenomenon, but these data do not help to identify why that phenomenon has that feature. One example of this situation is the discovery of Poiseuille's law.[1] The French scientist Jean-Louis Poiseuille (1799-1869) made extensive measurements of the flow of water through thin glass tubes of various diameters and lengths. In his 1846 report Poiseuille used these measurements to justify the claim that

$$Q = ka^4 \frac{(p_1 - p_2)}{l} \tag{1}$$

Q is here the rate of flow of the fluid, $(p_1 - p_2)$ is the pressure drop across a tube of length l, a is the radius of the tube, and k is a parameter that varied with fluid and temperature [Sutera and Skalak, 1993]. Poiseuille carefully restricted his data to flows that occurred in certain laboratory conditions. While he achieved a high level of control over this phenomenon, Poiseuille could not explain why his law held when it did or clarify why it would fail to apply in a broader range of circumstances.

In his 1884 Presidential Address to the British Association for the Advancement of Science, Lord Rayleigh (John William Strutt, 1842-1919) emphasized how new scientific discoveries create the need for new theoretical explanations:

> If, as is sometimes supposed, science consisted in nothing but the laborious accumulation of facts, it would soon come to a stand-still, crushed, as it were,

[1] This law is sometimes called the Hagen-Poiseuille law due to Hagen's priority in discovering it. See section 3 for some discussion.

> under its own weight. The suggestion of a new idea, or the detection of a law, supersedes much that had previously been a burden upon the memory, and by introducing order and coherence facilitates the retention of the remainder in an available form ... Two processes are thus at work side by side, the reception of new material and the digestion and assimilation of the old; and as both are essential, we may spare ourselves the discussion of their relative importance. One remark, however, should be made. The work which deserves, but I am afraid does not always receive, the most credit is that in which discovery and explanation go hand in hand, in which not only are new facts presented, but their relation to old ones is pointed out ([Rayleigh, 1900], 351).

Although Rayleigh here mentions only Ohm's law, earlier in his address he had singled out fluid dynamics as a central area for future investigation. Poiseuille's law and its relations to theory are at the heart of these investigations. For even though "[t]he laws of motion in capillary tubes, discovered experimentally by Poiseuille, are in complete harmony with theory ... when we come to the larger pipes and higher velocities with which engineers usually have to deal, the theory which presupposes a regularly stratified motion evidently ceases to be applicable ..." ([Rayleigh, 1900], 344). Rayleigh here notes some recent work by Osborne Reynolds (1842-1912), who "has traced with much success the passage from the one state of things to the other ..." ([Rayleigh, 1900], 344). As we will see in more detail below, this transition from so-called laminar to turbulent flow coincides with the breakdown of Poiseuille's law, and has proven very difficult to understand. Rayleigh highlights its significance: "In spite of the difficulties which beset both the theoretical and experimental treatment, we may hope to attain before long to a better understanding of a subject which is certainly second to none in scientific as well as practical interest" ([Rayleigh, 1900], 344). In summary, the theoretical explanation of Poiseuille's law highlights a gap in our understanding of the scope of this law. In section 5 I consider the heuristic role of this explanation in Reynolds' experimental investigations of the onset of turbulent flow.

Olivier Darrigol draws on this last remark by Rayleigh in his masterful *Worlds of Flow: A History of Hydrodynamics from the Bernoullis to Prandtl* ([Darrigol, 2005], 211, 217). This book shows how the development of fluid dynamics in the nineteenth century is almost as intricate as the turbulent flow of a fluid, where promising insights are abandoned only to be rediscovered, and important practical aspects of fluids are dismissed by theorizers. This pattern is clear in scientists' shifting attitudes towards what we now treat as the basic equations of fluid dynamics, the Navier-Stokes equations. These equations provide a more realistic treatment of fluids by including terms that reflect the fluid's viscosity or internal friction. But scientists resisted the introduction of such terms based on a number of considerations, including the greater mathematical tractability of the Euler equations, which treat fluids as inviscid. In his chapter on viscosity, Darrigol shows that the Navier-Stokes equations were proposed by Navier himself in the 1820s, and later motivated in a different way by Stokes in the 1840s ([Darrigol, 2005], 117, 138). Some of the hesitation to adopt the Navier-Stokes equations is traced by Darrigol to difficulties in using it to understand the flow of fluids through pipes. As Darrigol puts it,

> as late as 1860, the Navier-Stokes equation did not yet belong to the

> physicist's standard toolbox. It could still be rediscovered. The boundary condition, which is crucial in judging consequences for fluid resistance and flow retardation, was still a matter of discussion. Nearly twenty years elapsed before Horace Lamb judged the Navier-Stokes equation and Stokes's boundary condition to be worth a chapter in a treatise on hydrodynamics. This evolution rested on the few successes met in the ideal circumstances of slow or small-scale motion, and on the confirmation of the equation by Maxwell's kinetic theory of gases in 1866. Until Reynolds's and Boussinesq's studies of turbulent flow in the 1880s . . . the equation remained completely irrelevant to hydraulics ([Darrigol, 2005], 144).

In the next section and section 3, I will review some of the elements of this complicated history, as Darrigol develops it. This will pave the way for section 4 where I will consider Lamb's discussions of Poiseuille's law in his 1879 textbook and how that discussion was altered for the 1895 edition of that textbook. We will see Lamb's increased confidence in 1895 that his derivation amounts to an explanation of Poiseuille's law.

The philosophical payoff of these historical discussions concerns the character and value of scientific explanation, especially when that explanation is highly mathematical. In section 5 I will draw three lessons about explanation from this case. First, a derivation can count as an explanation even when that derivation includes idealizations or simplifications. Second, these explanations can be endorsed by Woodward's interventionist approach to causal explanation. Third, as Rayleigh suggests, there are important cases where "discovery and explanation go hand in hand" and where achieving an explanation has heuristic value in facilitating new, important discoveries.[2]

[2]Another episode of this sort is emphasized by Heidelberger: early in the twentieth century Prandtl

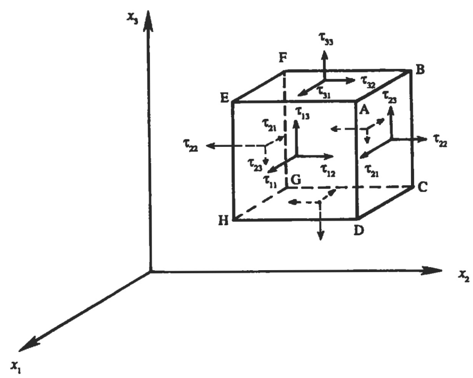

Figure 1: Normal and tangential stresses (Kundu and Cohen 2008, 31)

2. Historical Overview. In his 1879 *Treatise on the Mathematical Theory of the Motion of Fluids* Lamb begins with a derivation of the so-called Euler equations for fluid motion. As Darrigol notes, these equations were derived by Euler in the 18th-century by considering how a cubic element of a fluid would change through an external force like gravity and internal forces arising from pressure differentials on the sides of that fluid element ([Darrigol, 2005], 24). To appreciate how such derivations work, consider a cubic element of fluid aligned with the axes x_1, x_2 and x_3 (figure 1). A *stress* is a force divided by the area over which the force is applied. In principle, the fluid element will change due to both *normal* stresses that are applied in a direction that is perpendicular to a side of the fluid element as well as *tangential* stresses that are applied in a direction that is parallel to a side of the fluid element. Here τ_{11}, τ_{22} and τ_{33} are the normal stresses, while the other arrows represent tangential stresses. Lamb, following Euler and others in this tradition, observes that "It is usual, however, in the first instance, to neglect the

"found a way to bring together the purely empirical engineering tradition of hydraulics and the purely theoretical mathematical tradition of rational mechanics as it had developed in the 18th century" ([Heidelberger, 2006], 50). See also ([Darrigol, 2005], ch. 7) for more on Prandtl as well as [Darrigol, 2008] for Darrigol's discussion of the philosophical import of this history.

tangential stresses altogether" ([Lamb, 1879], 2). With this simplification, it is possible to identify the stresses applied to a fluid element with the differences in the scalar pressure field across the faces of the element.[3] The changes in velocity of a fluid element due to internal forces will then be entirely due to the pressure gradients across the fluid element in various directions: a higher pressure on one side will correspond to a push on that side. Pressure was also a quantity that could be readily measured.

The basic idea behind the Euler equations is that, in line with Newton's second law, the acceleration that a fluid element undergoes must be equal to the forces applied to that element divided by their mass. On the left side of these equations we collect all the terms that reflect that acceleration or change in the velocity of the fluid element. On the right side of these equations, we present each of the external and internal forces involved. In Lamb's notation, for spatial directions x, y, z, one such equation is[4]

$$\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz} = X - \frac{1}{\rho}\frac{dp}{dx} \tag{2}$$

Here u is the velocity of the fluid element in the x direction, X is the x component of the externally impressed force (such as gravity) (per unit mass), ρ is the density of the fluid and p is the pressure. Two other similar equations are available for v, the velocity of the fluid element in the y direction, and w, the velocity of the fluid element in the z direction.

These three equations form the core of the mathematical theory of an ideal fluid. We restrict our focus to a fluid whose density is a constant. The conservation of mass

[3] [Lamb, 1879], 2-3. See [Kundu and Cohen, 2008], 9-10 for a more thorough contemporary treatment.
[4] [Lamb, 1879], 5.

requires

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \tag{3}$$

These four equations characterize our five scalar quantities u, v, w, p and ρ throughout the interior of the fluid. Once additional equations that cover the boundary of the flow region are provided, a well-formed mathematical problem results. Many sorts of fluid flow could be treated in this way, but for flows near solid boundaries, there was a manifest disconnect between what the theory would predict and what measurements showed.

As Darrigol explains in his book, scientists quickly realized that the Euler equations could not be used in any straightforward way to explain many observed cases of fluid motion. In retrospect, it might seem obvious that one way forward would be to amend Euler's equations to take account of the tangential stresses applied to a fluid element in addition to the normal stresses. However, a number of scientists explored other options. For example, Helmholtz focused on vortex motion as the key to analyzing the effects of the internal friction or viscosity of a fluid ([Darrigol, 2005], 148-159). If these rotations in the fluid arose, how would they evolve and what effects would they have? These investigations bypassed the tricky question of how vortex motion would arise. It seemed natural to assume that a slightly viscous fluid would generate these vortices as it flowed, but the mechanism for this generation remained unclear. Given this mystery, one can appreciate why these researchers were reluctant to endorse the shift to the Navier-Stokes equation approach. It seemed to abandon the promise of what had been achieved with little assurance of success.

When Lamb comes to derive the Navier-Stokes equations in the last chapter of his 1879 *Treatise*, he notes that they have been derived by "Navier, Cauchy, Poisson, and others, on various considerations as to the nature and mutual action of the ultimate molecules of fluids" ([Lamb, 1879], 221). By contrast, "The method adopted above, which seems due in principle to de Saint-Venant and Stokes, is independent of all hypotheses of this kind ...". Still, Lamb continues, "it must be remembered that it involves the assumption that $p_{xx} + p$, p_{xy}, &c. are *linear* functions of the coefficients of distortion. Hence although (14) and (15) [the Navier-Stokes equations for compressible and incompressible fluids, respectively] may apply with great accuracy to cases of slow motion, [footnote] we have no assurance of their validity in other cases" ([Lamb, 1879], 221). The footnote refers to experiments by Maxwell as well as Helmholtz and Piotrowski. Lamb goes on to mention Maxwell's 1867 paper on the kinetic theory of gases, which validated the Navier-Stokes equations in a dilute gas.

Lamb's summary aligns quite well with Darrigol's more thorough historical treatment of viscosity ([Darrigol, 2005], ch. 3). On the one hand, it seemed necessary to build up an account of viscous fluid motion by considering how fluid molecules would interact. On the other hand, the forces governing this interaction were unknown and so their employment could seem too speculative. Saint-Venant and Stokes tried to avoid these worries by characterizing viscosity in non-molecular terms via the tangential stresses on a fluid element. However, even this approach required an assumption of linearity that Lamb at least worried would restrict the scope of the resulting equations.

To appreciate these concerns, we should return to figure 1. Here two tangential stresses are τ_{13} and τ_{12}. They involve stresses along the surface normal to the x_1 axis in the directions x_3 and x_2, respectively. These stresses would go along with distortions in the shape of the fluid element, as when a cube is shifted over to form a parallelepiped (figure 2). Following Stokes, Lamb supposed that the tangential stresses would be linear

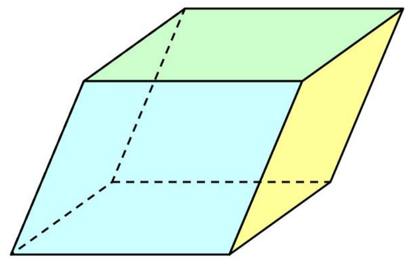

Figure 2: Parallelepiped

functions of the changes in velocities across the fluid elements.[5] Lamb's idea seems to be that these velocity gradients would be small for a small fluid element, at least in certain situations. One case that we will focus on is when the change in velocity u occurs over a distance a like the radius of a tube. If the ratio u/a was small enough, the function in question could be treated as if it were linear. These assumptions allow Lamb to introduce a new constant μ and a new term into his equations for an incompressible fluid ([Lamb, 1879], 221):

$$\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz} = X - \frac{1}{\rho}\frac{dp}{dx} + \frac{\mu}{\rho}(\frac{d^2u}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2w}{dz^2}) \tag{4}$$

Similar equations are available for v and w, with (3) staying the same. But what does μ reflect? μ, the coefficient of viscosity, is a kind of friction: if a fluid is "moving in a series of horizontal planes" ([Lamb, 1879], 219), one plane may be moving more slowly than its neighbor. The slower plane will exert a retarding stress on the faster plane that is proportional to the difference in velocity times the constant μ.[6] μ is here assumed to be

[5]See [Darrigol, 2005], 135-140 for discussion of Stokes.
[6]See [Kundu and Cohen, 2008], 6-8, 100-104 for some discussion.

an intrinsic feature of a fluid (at a given temperature), like its density ρ.

In summary, then, both the derivation of the Euler equations and the derivation of Navier-Stokes equations were contentious. The Euler equations ignore tangential stresses and this seemed to be partly responsible for their failure to capture a wide range of fluid behavior. The Navier-Stokes equations did incorporate tangential stresses, but only via assumptions about the linear character of certain terms. As these terms could not be independently evaluated, a failure to apply the Navier-Stokes equations could be blamed on this simplification. Similar issues arose for the boundary conditions that were needed to use the Navier-Stokes equations in any concrete case: if the theoretical predications failed to match the experimental measurements, then one could blame either the equations or the boundary conditions. Until some additional progress was made on these questions, it seemed prudent to remain cautious.

3. Poiseuille's Law. Poiseuille's scientific education began with a year at the famed Ecole Polytechnique (1815-1816), where he studied with Cauchy, Ampére and Petit, among others ([Sutera and Skalak, 1993], 1). Poiseuille later turned to the study of medicine, although his work retains the emphasis on precision and measurement characteristic of much of French science during this period.[7] One major innovation of Poiseuille's doctoral work *The force of the aortic heart* was a new device for measuring blood pressure. It was this focus on blood circulation or "hemodynamics" that later motivated Poiseuille to investigate the flow of fluids through tubes. Glass tubes with diameters from .015 to .6 mm were attached to a carefully constructed experimental apparatus, and distilled water was driven through the tubes across a range of controlled pressures. Poiseuille's 1846 report on these experiments is over 100 pages long, and

[7]See, e.g., the discussion of Regnault in [Chang, 2004], 96-102.

includes a detailed description of the way the experiment was conducted and the resulting data. Equation (1) was obtained through a careful examination of this data (see esp. §111 (512) and §129 (519) of [Poiseuille, 1846]).

Equation (1) proved to be remarkably wide in scope, applying not only to the flow of water at various temperatures, but also to other fluids, and to tubes composed of materials beside glass. However, knowing that the law held did not settle why the law held. Poiseuille showed no interest in explaining his law using the purportedly more basic equations for flow through pipes that Navier had formulated in the 1820s. As Darrigol notes, Poiseuille "only mentioned Navier's theory to condemn it for leading to the wrong PR^3/L law" ([Darrigol, 2005], 143). In his report Poiseuille emphasizes the need to proceed empirically, without the use of Navier's unwarranted "hypotheses" concerning the interactions between fluid "molécules" ([Poiseuille, 1846], 435).

Darrigol notes how Poiseuille's law was discovered before Poiseuille by the engineer Gotthilf Hagen (1797-1884) ([Darrigol, 2005], 140). In 1839, Hagen used pipes that had a larger diameter than Poiseuille's, but were still quite thin, with diameters between 1 and 3 mm. Hagen obtained a formula that included a term for the "entrance effect" that arises when the fluid enters the tube. Hagen also noted that his formula would fail to apply beyond a certain threshold velocity. Darrigol concludes that "Hagen's priority and the excellence of his experimental method are undeniable" ([Darrigol, 2005], 141).

Prior to Lamb's 1879 textbook, a number of authors had proposed a derivation of Poiseuille's law using the Navier-Stokes equation. Darrigol notes that "Helmholtz was probably the first physicist to link the Navier-Stokes equation to the Hagen-Poiseuille law" ([Darrigol, 2005], 143). In addition, Franz Neumann derived the law from the Navier-Stokes equation by assuming a no slip boundary condition, which Darrigol reports was published by his student Jacobson in 1860. Additional proofs were provided by Hagenbach in 1860 and Mathieu in 1863 ([Darrigol, 2005], 143).

4. Lamb's 1879 *Treatise* vs. 1895 *Hydrodynamics*. Horace Lamb (1849-1934) was a British physicist, trained at Cambridge, who prized the principled, theoretical treatment of scientific phenomena. His 1879 textbook emphasized the importance of theory from the beginning: it is an "attempt to set forth in a systematic and connected form the present state of the theory of the Motion of Fluids" ([Lamb, 1879], v). In a later address in 1904 Lamb emphasized the need for this kind systematic treatment, in line with Rayleigh's 1884 address: "It is ... essential that from time to time someone should come forward to sort out and arrange the accumulated material, rejecting what has proved unimportant, and welding the rest into a connected system" (given at [Love and Glazebrook, 1935], 384). As the title of the 1879 book indicates, for Lamb the right way to present things in a "systematic and connected" way is through a "mathematical theory". When Lamb revised and expanded this book in 1895 he gave it the new title *Hydrodynamics*. This textbook became the standard reference in English for the theory of fluid mechanics.

As noted already, nearly all of Lamb's 1879 book relates to the theory of ideal fluids where the main tool is the Euler equations. These equations are manifestly unable to derive Poiseuille's law. One way to appreciate the situation is to note that the conservation of momentum requires that there is no pressure drop along the pipe as the fluid flows through it. So if the velocity profile is uniform when the fluid enters the pipe, it will continue to be uniform as it flows through the pipe. The rate of flow Q through a circular pipe is the integral of the velocity profile across the area of section of the pipe. For a constant velocity profile, this means that $Q \propto a^2$. So, for example, doubling the radius of the pipe from 1 cm to 2 cm should quadruple the rate of flow Q. But (1) maintains that $Q \propto a^4$, so that doubling the radius of the pipe from 1 cm to 2 cm should increase the rate of flow Q by a factor of 16. Poiseuille's measurements of pressure showed that the pressure was lower at one end of the pipe than the other. The flow exhibited a kind of resistance that mandated a higher pressure at one end for the flow to be steady.

In his 1879 textbook, right after deriving the Navier-Stokes equations, Lamb applies them to the case of Poiseuille's fluid flow through a circular pipe ([Lamb, 1879], 223-224).[8] One crucial issue for such a flow is what one should suppose is going on at the boundary between the fluid and the pipe. Now that one plane of fluid could slow down another due to viscosity, it was natural to suppose that the unmoving walls of a pipe would further retard the flow. In his 1879 discussion Lamb notes that some experiments had found 0 velocity at the boundary (or "no slip"), while others had detected some finite velocity (or "slip") at the boundary. Lamb deals with this disagreement by introducing a "coefficient of friction" β: as β gets bigger, the velocity is decreased more rapidly by the friction between the wall and the fluid at the boundary. For the pipe flow case, where r measures the radial direction away from the center of the pipe, the boundary condition becomes

$$-\frac{du}{dr} = \frac{\beta}{\mu} u \tag{5}$$

for $r = a$, the radius of the pipe. Additional simplifications arise for Poiseuille's flow once we assume that it is a special sort of steady flow: all velocities except those in the x direction are 0. The original continuity equation can thus be simplified to $\frac{du}{dx} = 0$. This

[8] See [Kundu and Cohen, 2008], 302-303, [Milnor, 1989], 11-17.

allows the decomposition of the flow into thin shells or laminae, where the velocity in a given shell is constant as we move down the pipe in the x direction. Lamb solves the simplified system of differential equations so that

$$u = \frac{1}{4}A(r^2 - a^2) - \frac{1}{2}\frac{\mu a}{\beta}A$$

where the constant A is identified with a term that involves the length of the pipe l and the pressure drop across the pipe $(p_1 - p_2)$:

$$A = -\frac{(p_1 - p_2)}{\mu l}$$

The velocity profile of the flow is thus a parabola whose apex is at the center of the pipe $(r = 0)$. To obtain the rate of flow of the fluid through the pipe (volume/time), Lamb takes the spatial integral across a circular section of the pipe, obtaining

$$Q = \frac{1}{8}\frac{\pi}{\mu}a^4\frac{(p_1 - p_2)}{l} + \frac{1}{2}\frac{\pi}{\beta}a^3\frac{(p_1 - p_2)}{l} \tag{6}$$

Lamb notes that if no slip occurs at the boundary (β goes to infinity),

$$Q = \frac{1}{8}\frac{\pi}{\mu}a^4\frac{(p_1 - p_2)}{l} \tag{7}$$

This flow equation is Poiseuille's law (1), with $k = (1/8)\,\pi/\mu$. He adds that a comparison of (7) "with experiments of this kind would give the means of determining μ" ([Lamb, 1879], 224). In these cases, then, one obtains a way to measure the coefficient of viscosity μ.

Although we have skipped over the mathematical subtleties, the theoretical significance of this derivation should be clear. By deploying the Navier-Stokes equations for viscous fluids, Lamb was able to theoretically derive (1). This shows that Poiseuille's k, which he found to vary with temperature, can be decomposed into a constant times Lamb's viscosity μ, which had been independently found to vary with temperature. In addition, if we accept this derivation, we can make sense of the cases where Poiseuille's law holds and also diagnose two factors that would lead it to break down. For the law to hold, the velocity of the fluid must go to zero at the boundary so that no slip occurs. The law would then break down when some finite velocity of the flow occurred at the boundary, as measured by Lamb's coefficient of friction β. A more basic condition, tied to the Navier-Stokes equations, is that the velocity profile of the flow must be in the shape of a parabola, and not a constant. This is the way to recover the dependence between the rate of flow and the fourth power of the radius of the pipe. The observed resistance of the fluid to flowing through the pipe can be attributed to the viscosity of the fluid.

It is fair to say that in 1879 Lamb was cautious about this particular derivation. He recognized that he would have an explanation of Poiseuille's law if the assumptions of the derivation held for real fluid flows through circular tubes. But the scope of these assumptions remained unclear. However, by 1895 the situation had changed considerably. As we will see, in 1895 Lamb is now very confident in the no slip boundary condition and in the treatment of μ. One major reason for this shift is Reynolds' work in the 1880s. Reynolds' groundbreaking paper on fluid flow through pipes is "An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels" from 1883. Some of

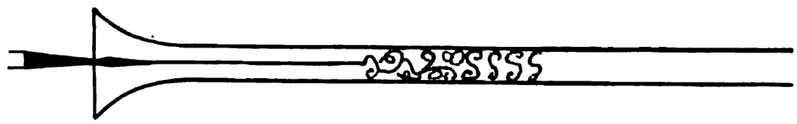

Figure 3: Transition to turbulent flow (Reynolds 1901, 60)

the lessons of these experiments for Poiseuille's law are clarified in "On the theory of lubrication and its application to Mr. Beauchamp Tower's experiments, including an experimental determination of the viscosity of olive oil" from 1886.[9] The 1883 paper begins by noting two "leading features" that were thought to obtain for various fluid flows. First, some fluid flows can be decomposed into straight lines, while others "eddy about in sinuous paths the most indirect possible" ([Reynolds, 1901], 52). Second, for some fluid flows, the resistance is proportional to the fluid velocity, while for others the resistance is proportional to the square of the fluid velocity. If the resistance is not proportional to the fluid velocity, but to the square of the velocity (or some other power much greater than 1), then a given pressure gradient will generate much less fluid flow.

Reynolds' first innovation was to refine the means to observe the formation of eddies in fluids through the use of colored dyes. He first constructed an experimental apparatus with a transparent tube that would allow one to observe the straight flow of the colored water as it joined the flow of water through the tube. Then as the pressure differential was increased, and the fluid velocity also increased, he could record the precise circumstances under which the direct or "laminar" flow would break down. At a certain point the flow would exhibit eddies that showed a kind of circular or vortex motion (figure 3). This shift, to what became known as "turbulent" flow, coincided with the

[9] All page references to these papers are from the reprinted versions in [Reynolds, 1901].

break down in the direct relation between resistance and velocity. The addition of eddies disrupted the smooth flow and required greater pressure differences to sustain a given rate of fluid flow.

The transition from laminar to turbulent flow should be somehow tied to the viscosity of the fluid. Reynolds supposed that the laminar flow was more or less unstable, but that a laminar flow could be maintained if the viscosity of the fluid dampened any nascent eddying motion of the fluid. However, at some point, this damping tendency would be overcome, and the flow would become turbulent. Despite this insight, Reynolds lacked any direct theoretical treatment of the transition for one obvious reason: the mathematics of the Navier-Stokes equations was too complicated to handle except for special cases like the steady, laminar flows that Lamb had supposed in his derivation of (7). The transition from laminar to turbulent flow involves an especially tricky case of unsteady flow, where some dramatic changes occur in the velocities of the fluid elements. So Reynolds was forced to rely on other methods to understand how turbulent flow could arise.

The failure to explain the breakdown of laminar fluid flow suggested to Reynolds some broader gaps in our understanding: "This accidental fitness of the theory to explain certain phenomena while entirely failing to explain others, affords strong presumption that there are some fundamental principles of fluid motion of which due account has not been taken in the theory" ([Reynolds, 1901], 53). Reynolds' major theoretical innovation was to use the Navier-Stokes equations to arrive at a ratio that he argued should mark the transition from laminar to turbulent flow. This ratio is now called the Reynolds number: $\frac{c\rho U}{\mu}$ ([Reynolds, 1901], 55). c is some length like the radius of the pipe, ρ is the density, U is a velocity and μ is the viscosity. Reynolds proposed that the transition from laminar to turbulent flow occurred when this number exceeded some threshold, which he tried to determine experimentally.[10] After examining a wide range of flows that systematically varied the radius c, the velocity U and the viscosity μ, Reynolds showed how the relationship between resistance and velocity shifted in the same way at the same critical point.[11] As he summarized these results in his 1886 paper, for circular tubes the proposed critical value for this parameter was 1400 ([Reynolds, 1901], 238). Below this value, the flow would be direct and conform to Poiseuille's law, while above this value the flow would be turbulent and Poiseuille's law would no longer apply.

For our purposes, the main achievements of this work by Reynolds are tied to how they provided a kind of validation of Lamb's treatment of viscosity. To start, Reynolds emphasized how widely (7) obtained, for tubes of various sizes and composition, and for a wide range of pressures and fluid velocities: in a first series of experiments "Up to these critical velocities the discharge from the pipes agreed exactly with those given by Poiseuille's formula for capillary tubes" ([Reynolds, 1901], 64), and in a second series of experiments, "it is a matter of no small interest to find that the formula of discharges which he [Poiseuille] obtained from these experiments [as reported in [Poiseuille, 1846]] is numerically exact for the bright metal tubes 100 times as large" ([Reynolds, 1901], 92).

Two conclusions seemed warranted. First, there was strong experimental evidence in favor of the "no slip" boundary condition, as any slip at the boundary would introduce some systematic deviations from what (7) required. Second, Lamb's focus on the magnitude of the velocities in his treatment of viscosity could be shown to be

[10]See [Langhaar, 1951], 24 for a modern reconstruction of Reynolds' reasoning. However Darrigol discusses the gap between modern dimensional analysis and Reynolds' discussion, noting that the "legend" of Reynolds' reasoning began with Stokes ([Darrigol, 2005], pp. 255-258).

[11]Reynolds varied the viscosity by varying the temperature of the fluid.

fundamentally misguided. Early in his 1886 paper Reynolds considers two different theoretical treatments of viscosity under the heading "The Two Viscosities" ([Reynolds, 1901], 236). Reynolds cites Lamb and notes that "it has been supposed that μ varied with the rate of distortion – i.e., is a function of u/a, but is sensibly constant when u/a is small" ([Reynolds, 1901], 236). The motivation for this treatment is that in certain circumstances a constant viscosity can be used to explain the character of the fluid flow, as in Lamb's derivation of (7). But in other situations, where resistance varies as the square of the velocity, μ cannot be treated as a constant "unless a restricted meaning be given to the definition of viscosity, excluding such part of the resistance as may be due, in the way explained by Prof. Stokes, to internal eddies or cross streams ..." ([Reynolds, 1901], 236). Armed with his analysis of the transition from laminar to turbulent flow, Reynolds argues that we should treat viscosity as a constant for a fluid at a given temperature. Lamb's attempt to limit the scope of the Navier-Stokes equations in terms of a ratio like u/a is thus misguided. There are cases where the ratio between u and a, the radius of the tube, is small, and yet (1) fails, and also cases where the ratio is very large, and yet (1) holds. The true parameter that distinguishes these cases is $\frac{au\rho}{\mu}$, which takes the viscosity μ to be a "physical property of the fluid which is independent of its motion" ([Reynolds, 1901], 237).

Although there were many developments in fluid dynamics between 1879 and 1895, when we consider Lamb's 1895 treatment of Poiseuille's law, it seems clear that Reynolds' work played a major role in Lamb's revisions. One symptom of this change is that Lamb is now very confident in his no slip boundary condition and also in his linear treatment of tangential stresses. After presenting his derivation of Poiseuille's law from the Navier-Stokes equations, Lamb now adds that "This last result is of great importance as furnishing a conclusive proof that there is in these experiments no appreciable slipping of the fluid in contact with the wall" ([Lamb, 1895], 521). This is because, for the quantities at issue, "a deviation from the law of the fourth power of the diameter, which was found to hold very exactly, would become apparent" ([Lamb, 1895], 522). Here Lamb cites an 1890 paper by Whetham, who tested Poiseuille's law for tubes made up of materials besides glass [Whetham, 1890].[12] This additional experimental work seems to have played an important part in convincing Lamb that the no slip condition applied quite widely. Lamb thus rejects the results of Helmholtz and Piotrowski that indicated some finite slip velocity.

However, some more significant changes are tied to Reynolds' treatment of laminar flow. In his 1895 derivation of the Navier-Stokes equations, Lamb now initially motivates his linear approach to tangential stresses via simplicity: "The simplest hypothesis we can frame on this point is that these functions are linear" ([Lamb, 1895], 511). After deriving the equations, he repeats that his hypothesis "is of a purely tentative character . . . we have so far no assurance that it will hold generally" ([Lamb, 1895], 513). Lamb continues, though, by noting the argument from the 1886 paper by Reynolds:

> It has however been pointed out by Prof. Osborne Reynolds that the equations based on this hypothesis have been put to a very severe test in the experiments of Poiseuille and others . . . Considering the very wide range of values of the rates of distortion over which these experiments extend, we can hardly hesitate to accept the equations in question as a complete statement of the laws of viscosity ([Lamb, 1895], 513).[13]

[12] This shift in Lamb's treatment of the no slip boundary condition is noted by [Day, 1990].

[13] Lamb also here notes the work by Maxwell on the kinetic theory of gases that informed the note to

As we have seen, Poiseuille's law breaks down in certain special circumstances that Reynolds argued were tied to the Reynolds number. This convinced Lamb that the treatment of viscosity in terms of μ was legitimate, even when the ratio u/a was not small. For if one could only treat viscosity in terms of μ when this ratio small, then the Navier-Stokes equations would not apply to fluid flow through pipes when the ratio was not small. And if the Navier-Stokes equations failed for such a flow, then the flow would not conform to Poiseuille's law. But Reynolds found that Poiseuille's law in fact applied to many high velocity flows. So it is no surprise to find that Lamb changes his attitude towards the use of Poiseuille's law to measure viscosity. Whereas in 1879 he said only that "A comparison of the formula ([7]) with experiments of this kind would give the means of determining μ", in 1895 Lamb writes that "The assumption of no slipping being thus justified, the comparison of the formula ([7]) with experiment gives a very direct means of determining the value of the coefficient μ for various fluids" ([Lamb, 1895], 522).

In summary, Lamb's treatment of tangential stresses in terms of a constant μ was initially a mathematical simplification with little theoretical basis. After Reynolds' work, there was still no satisfying theoretical explanation for why μ was a constant or how a given value for μ could be determined by the more basic characteristics of the fluid molecules. However, what Reynolds offered was experimental support for Lamb's treatment of viscosity. If we add to our theory of these fluids that μ is a physical property of the fluid (at that temperature), then we can explain the wide range of application of (1). In addition, we have the beginnings of an understanding of turbulent flow based on Reynold's criterion for when turbulent flow would arise.

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the 1879 edition that I discuss below in section 5. This note, however, does not appear in the 1895 edition.

5. Explanation and Idealization. The work by Lamb and Reynolds that we have reviewed involves an intricate interplay of experimental and theoretical considerations. From a philosophical perspective, perhaps the most striking aspect of their discussions of Poiseuille's law is the prominent use of explanatory language. In this section I offer a reconstruction of their reasoning that involves three claims. First, to make sense of when a derivation that incorporates an idealization should count as an explanation, we must consider the subject matter of the derivation. Second, these derivations can be used to provide causal explanations using the interventionist account developed by Woodward. Third, when these explanations are accepted, they have additional heuristic benefits in facilitating new and important discoveries.

It seems that practitioners like Rayleigh, Lamb and Reynolds identify an explanation of Poiseuille's law with a derivation from the Navier-Stokes equations. When these equations are supplemented by the no slip boundary condition and additional assumptions like the steadiness of the flow, it is fairly straightforward to derive this law. However, this does not settle what makes this derivation explanatory. So we should consider what makes some derivations explanatory, especially when idealizations are involved.

Most recent philosophical discussions of scientific explanation begin with Hempel and Oppenheim's famous "deductive-nomological" (D-N) account of explanation.[14] On this account, an explanation consists in a deductively valid argument whose conclusion is the statement to be explained. This argument must include at least one scientific law and all of the premises must be true.[15] It might seem like a D-N approach is in a good position

[14] I am grateful to an anonymous referee for urging me to make clearer the links between this case and the accounts of explanation offered by Hempel and Kitcher.

[15] See [Hempel, 1965], ch. 10, 12 for classic discussion as well as [Salmon, 1989] for some now classic

to make sense of how the Navier-Stokes equations explain Poiseuille's law through the derivation of that law from the equations. However, there is a well-known problem with the D-N approach when it is used to make sense of how one law can be used to explain another law. This problem was conceded by Hempel and Oppenheim in what Salmon calls a "notorious" footnote. As Hempel and Oppenheim put the worry:

> The precise rational reconstruction of explanation as applied to general regularities presents peculiar problems for which we can offer no solution at present ... The problem ... arises of setting up clear-cut criteria for the distinction of levels of explanation or for a comparison of generalized sentences as to their comprehensiveness. The establishment of adequate criteria for this purpose is as yet an open problem ([Hempel, 1965], 273).

As Hempel and Oppenheim sought to characterize this relationship in formal terms, they found this problem to be intractable. It is not clear how to make the Navier-Stokes equations or Poiseuille's law count as genuine laws on this approach, or what would ensure that the former was in an appropriately distinct level than the latter.

Arguably, the same sort of problem arises for other approaches that try to single out a special sort of derivation as an explanation. Kitcher has claimed that a derivation is an explanation when that derivation is an instance of a schema that is widely instantiated [Kitcher, 1989]. The unifying power of the argument schema makes its instances explain their conclusions. Whatever the merits of this approach in other cases, it is difficult to see how it can work for our derivation of Poiseuille's law. The specific steps in the derivation are tailored to the derivation of that very law. So if Kitcher wants to

objections.

emphasize these specific steps, then he will find that this derivation is not widely instantiated, and so not explanatory. Of course, Kitcher could respond that he means to consider any derivation that begins with the Navier-Stokes equations and ends with some law that is to be explained. The so-called "filling instructions" for such a derivation would thus be quite flexible and open-ended. One worry for this proposal is that there is no longer any illuminating characterization of the argument schema. The source of the explanatory power of the instances is no longer clear.

More recently, Baron has tried to retain the basic idea that some derivations are explanations by imposing additional conditions tied to relevance logic [Baron, 2019]. Baron aims here to pin down what is special about "mathematical explanations" as opposed to ordinary scientific explanations that use mathematics. That contrast is not my focus here.[16] What is interesting about Baron's proposal for this paper, though, is that it identifies some explanations with "sound $\mathcal{R}$-arguments, where an $\mathcal{R}$-argument is an argument in which all of the information contained within the conclusion of the argument is contained in the premises, and each premise contributes some part of the information contained within the conclusion" ([Baron, 2019], 712). One could appeal to "information" to identify when a given derivation is explanatory, and it might seem like this condition marks an improvement over Hempel, Oppenheim and Kitcher. Arguably, all the information contained in Poiseuille's law is to be found in the premises of the derivation. This is because we have a genuine derivation where the truth of the premises would guarantee the truth of the conclusion. In addition, each premise offered by Lamb seems to contribute some of that information.

One worry about Baron's proposal is that there are many derivations that meet this

[16] See [Bangu, 2020] for another recent contribution to this debate.

informational condition and yet fail to explain. Consider, for example, a small-scale concrete model of a ship. Information about the drag that this model ship experiences in the laboratory along with appropriate scaling laws is sufficient to derive the drag that a full-scale version of the ship experiences. As we have a derivation, all the information contained in the conclusion is found in the premises. Also, each premise contributes some of that information. But it is quite plausible that this derivation does not explain the drag experienced by the full-scale version of the ship. As we might put it, the derivation indicates what drag would be experienced, but not why that drag would occur. Baron could reply that this objection ignores some of the other conditions that he imposes on his explanatory derivations. These seem to relate to what would make such an explanation into a mathematical explanation, and so it is hard to see their significance. For example, Baron says that the notion of information he is working with is "semantic information" and that his theory aims to "provide a partial account of what kind of information explanatory information is: it is information that features in relationships of informational containment where those relationships involve mathematical facts essentially" ([Baron, 2019], 701). If we ignore the question of when such a derivation is a specifically mathematical explanation, then all we are left with is the condition of "informational containment". As informational containment is too weak a condition for a derivation to count as an explanation, it is hard to see why a derivation that meets the informational containment test through an essential appeal to a mathematical fact is thereby explanatory.

One common reaction to these problems is that one should not focus on *how* information about the conclusion is provided, but instead on a *special sort* of information, no matter how it is provided. Woodward in particular has argued that a derivation that provides the right kind of causal information about its conclusion should count as an explanation. According to Woodward a causal explanation includes a causal generalization along with a specification of the actual values of the causal variables mentioned in that generalization [Woodward, 2003]. A generalization is a causal generalization when it indicates how an intervention that changes the values of some variable would result in a change in some other variable. For example, both the Navier-Stokes equations and Poiseuille's law count as causal generalizations. Equation (4) indicates that an intervention on the viscosity μ of some fluid would change the value of the left-hand side of the equation that tracks the acceleration of the fluid element. Similarly, as emphasized earlier, (1) indicates that doubling the radius of the pipe a through an intervention would increase the flow rate Q by a factor of 16. How, though, can a derivation of (1) using (4) count as a Woodward-style causal explanation? The derivation is explanatory just in case it identifies interventions on causal variables that would disrupt the law being explained, here (1). One clear way in which this occurs concerns the role of viscosity in (1). If the value of μ is changed or even set to 0 through an intervention, then this would disrupt a fluid flow that conformed to (1). As we have seen, setting $\mu = 0$ would render the fluid inviscid. So if the initial entry velocity profile of the flow was uniform, the parabolic velocity profile would become uniform. The derivation indicates, then, how the law being explained could be disrupted.

A major issue for this sort of explanation is that the derivation includes false claims that Lamb and others are aware are false. I will call such a claim an idealization. Idealized derivations pose a problem for all the views we have considered, so it is worth considering if Woodward's approach has any special advantage on this front. Before turning to Woodward, I summarize an interesting discussion by Lamb in 1879 that goes some way to addressing this worry for the fluid dynamics case. Lamb begins his book with the following remark:

> The following investigations proceed on the assumption that the fluids with which we deal may be treated as practically continuous and homogeneous in structure; i.e. we assume that the properties of the smallest portions into which we can conceive them to be divided are the same as those of the substance in bulk. It is shewn in note (A), at the end of the book, that the fundamental equations arrived at on this supposition, with proper modifications of the meanings of the symbols, still hold when we take account of the heterogeneous or molecular structure which is most probably possessed by all ordinary matter ([Lamb, 1879], 1).[17]

If we can see how Lamb proposes to handle this "continuum" idealization, then we will be in a good position to consider how he might address any others that arise in his derivations.

Lamb's rough strategy is to defend what I will call a matching claim. For a derivation D of some theorem T that relies on an assumption A that is unjustified or believed to be false, the matching claim is that there is some corrected derivation D' of T that replaces A with some truth A'. As Lamb suggests, the corrected derivation can arise from a reinterpretation of the claims of the original derivation. To illustrate this idea, we can consider Lamb's derivation of the Euler equations. The matching claim for this derivation is that there is another derivation of those very same equations that correctly describes the molecular structure of fluids. One way to justify a matching claim

[17] The second sentence and the note are removed in the 1895 edition.

is to actually present this revised derivation. However, when the true character of fluids is unknown or too complicated to capture, an indirect justification of the matching claim is required. This is what Lamb offers in his note.

The key idea is that the fluid elements are large enough so that the fluctuating motions of individual molecules are insignificant: "We suppose the 'elements' above spoken of to be such that each of their dimensions is a large multiple of the average distance (d, say,) between the centres of inertia of neighboring molecules" ([Lamb, 1879], 232). When this condition is met, the dynamical behavior of the fluid elements in a real, discontinuous fluid will match the dynamical behavior of the fluid elements composed of a completely continuous and homogeneous stuff. As Lamb puts it after fleshing out how his terms are to be interpreted in terms of molecules, "The effect of the foregoing definitions is to replace the original (molecular) fluid by a *model*, made of an ideal continuous substance, in which only the main features of the motion are preserved" ([Lamb, 1879], 233, emphasis added). There is no rigorous mathematical proof that the correspondence is complete or that it will hold in all circumstances. However, Lamb takes himself to provide enough evidence of a match between the model and its target for the features of interest and fluid regimes that are his focus. Crucially, Lamb goes on to reexamine his treatment of tangential stresses in the derivation of the Navier-Stokes equations.[18] This involves comparing the actual stress on an element to the apparent stress. This apparent stress includes a correction for the transfer of molecules and their momenta across the element boundary. Lamb notes that "The apparent stress is what is really observed in all experiments in fluids. It is in fact the stress which must hold at any

[18]Lamb here notes that he is following Maxwell's 1867 paper "On the Dynamical Theory of Gases", which is reprinted in [Maxwell, 1890].

point of the continuous model above spoken of, in order that the model may work similarly to the original" ([Lamb, 1879], 236).

On this approach, an explanation of some real phenomenon can employ a mathematical model that fails to accurately represent the real phenomenon in significant respects. In Lamb's case, he admits that real fluids are composed of molecules, while his model fluids are continuous. However, the explanation is genuine so long as there is the right kind of match between the features exhibited by the mathematical model and the features of the real phenomenon.[19] Woodward's interventionist approach can be used to motivate one kind of matching that is sufficient for an idealized derivation to explain: the idealized derivation must correctly indicate some interventions that would disrupt the law being explained. As just noted for the derivation of Poiseuille's law, the relation $Q \propto a^4$ holds in both the mathematical model and the real phenomenon. In addition, the derivation indicates a reason for why $Q \propto a^4$ holds in the real phenomenon. Lamb's derivation proceeds through the Navier-Stokes equation, with its treatment of viscosity, and the no slip boundary condition. So if real fluids are viscous in the way required, and also exhibit no slip at the boundary, we can consider a steady flow that is decomposed into shells or laminae. If the entry profile is uniform, at some later point in the flow the velocity profile would be a parabola with $u = 0$ at the boundary. This is a genuine explanation for why $Q \propto a^4$. It is not required that real fluids be viscous in the way required at all spatial scales, but only for spatial scales corresponding to the size of Lamb's model elements. For as long as a match is obtained at this scale, both the real fluid and the model fluid will develop a parabolic velocity profile, and this profile will

[19]For some more general discussion of how idealized models can explain, see [Bokulich, 2017], [Potochnik, 2017] and [Pincock, 2021].

develop for the very same reasons. This indicates how the relation $Q \propto a^4$ that is being explained could be altered through an intervention.

It remains to discuss the heuristic benefits afforded by the explanation of Poiseuille's law using the Navier-Stokes equation along with the no-slip boundary condition. Schematically, we have a situation where some theory provides an explanation of one phenomenon, and a scientist takes this explanation for granted in the investigation of some other phenomenon. This explanation will provide a heuristic benefit to the extent that it facilitates the discovery of some new features of the other phenomenon. I suggest that there are two kinds of heuristic benefits to consider in such a case. First, if the two phenomena are thought to be appropriately related, then the scientist may be encouraged to pursue an explanation of the new phenomenon based on the accepted explanation of the old phenomenon. When some explanations are adopted, this makes scientists more confident that new phenomena also have explanations that are waiting to be discovered. Second, if the two phenomena are connected in an especially strong way, then the scientist may try to arrive at an explanation of the new phenomena by adapting the accepted explanation of the old phenomenon. The scientist supposes that there is some common explanatory framework that can be used to account for both the similarities and the differences between the two phenomena.

I claim that both sorts of heuristic benefit are present in Reynolds' investigation into the transition from laminar to turbulent flow. In this case, the wide scope of Poiseuille's law helped to establish the wide scope of the Navier-Stokes equation and the no-slip boundary condition. Reynolds accepted this explanation of Poiseuille's law. As noted earlier, he also used explanatory language to characterize the situation, and noted the contrast between the accepted explanations of some phenomena and the as-yet-unexplained character of other fluid phenomena: "This accidental fitness of the theory to explain certain phenomena while entirely failing to explain others, affords strong presumption that there are some fundamental principles of fluid motion of which due account has not been taken in the theory" ([Reynolds, 1901], 53). This passage indicates the first sort of heuristic benefit noted above: when Reynolds accepted the explanation of Poiseuille's law, he became more confident that the related phenomenon of the transition from laminar to tubulent flow would also have an explanation. He did not assume that the Navier-Stokes equations by themselves would be adequate to provide this explanation. He says that there may be "some fundamental principles" that are missing from the theory, and that would prove adequate to explain the transition. Still, it is clear that the Navier-Stokes equations were the starting point of Reynolds' theoretical investigations. By contrast, we can imagine a scientist who was not aware of the explanation of Poiseuille's law. They would not be similarly encouraged to pursue an explanation of the transition from laminar to turbulent flow based on the Navier-Stokes equations for the simple reason that it would be unclear if this transition had an explanation, or if these equations even applied to such cases.

The second kind of heuristic benefit involves trying to apply a common explanatory framework to the two phenomena. This is a plausible description of Reynolds' approach to the transition from laminar to turbulent flow. It is clear that the explanation of Poiseuille's law needs to be substantially changed in order to address this transition. In particular, an initial assumption of this explanation is that the flow can be decomposed into a series of thin shells that move at different velocities in the x direction. The onset of turbulent flow involves the disruption of this assumption. So the explanation of Poiseuille's law is powerless to indicate why such a disruption occurs. Still, the disruption of this assumption could provide Reynolds with a starting point for his reflections on this change. As noted above, he arrived at his criterion for the transition by considering the relative magnitude of the terms in the Navier-Stokes equations. His hunch was that the transition resulted from the "inertial" terms overcoming the damping provided by the "viscous" terms. If the relative magnitudes of these terms went along with the transition, then one would expect measurements of the ratio of their values for various instances of that transition to agree. This is what Reynolds found when he conducted his experiments. This experimental verification provided even more encouragement for Reynolds to pursue his theoretical account of the transition to turbulence.[20]

It is important to emphasize that a heuristic benefit obtains when the discovery of some new result is facilitated. By this I mean that the discovery was easier than it otherwise would have been. A heuristic benefit is thus neither essential nor sufficient to the discovery. Even without the explanation of Poiseuille's law, it would certainly have been possible for Reynolds to arrive at his criterion. I am only claiming that it was easier for him to arrive at this discovery once he had the explanation in hand. Also, once Reynolds had the explanation of Poiseuille's law, it was not guaranteed that he would have arrived at his criterion. It took additional insight and creativity to arrive at this proposal.

6. Conclusion. Schematically, the process that we have seen involves a fruitful interaction between experiment and theoretical explanation. An experimental law is identified, here Poiseuille's law. Then various theoretical explanations of that

[20]See [Launder, 2014] for some discussion of Lamb's reactions to Reynolds' theoretical innovations in an 1895 paper. Lamb emphasizes that the transition to turbulence is "the chief outstanding difficulty of our subject" ([Lamb, 1895], 572), and notes Reynolds' proposal without endorsing it ([Lamb, 1895], 579). See [Darrigol, 2005], 260-262 for some discussion.

experimental law are attempted. One explanation is then singled out for further scrutiny. If a threshold of evidence is reached, the otherwise contentious elements of this explanation may be accepted, in part because they afford a satisfying explanation. It then becomes feasible to exploit the theoretical treatment of the first experimental law to identify new opportunities for new experimental laws to be isolated. In our case, this new experimental law concerns the transition from laminar to turbulent flow based on the Reynolds number for the flow. The new experimental law then cries out for a new theoretical explanation, which may in turn validate new theoretical assumptions. In the case of turbulence, it is hard to exaggerate the theoretical significance of the transition that Reynolds sought to explain. It remains one of the most significant aspects of active investigation into fluid dynamics. Here, then, we have a case where mathematical derivations of experimental laws have been accorded significant scientific value as providing explanations. In addition, these explanations arguably have heuristic benefits for the ongoing process of scientific discovery.

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